Journal of Mathematical Analysis and Applications **254,** 101–119 (2001)
doi:10.1006/jmaa.2000.7210, available online at http://www.idealibrary.com on IDEAL®

On the Equation $\nabla u = uV$

Holger Teismann

Department of Mathematics and Statistics, University of Saskatchewan,

Submitted by B. S. Thomson

Received December 8, 1999

The PDE $\nabla u = uV$ can be treated as its ODE equivalent $u' = a\,u$ if the vector field V satisfies the integrability condition $\operatorname{curl} V = 0$. In this case there exist solutions v of the equation $\nabla v = V$ and solutions of the original equation have the form $u = ce^v$ ($c \in \mathbb{C}$). However, in devising meaningful solution concepts, the equation displays some surprising features related to local and global integrability of its solutions. In the course of the investigation technical tools like a lifting theorem with respect to the exponential map and Poincaré type theorems are proved for distributions and functions of Beppo Levi type.

1. INTRODUCTION

In this note we study some properties of the simple PDE

$$\nabla u(x) = u(x)V(x), \qquad x \in \mathbb{R}^n \tag{$*_n$}$$

($n > 1$) where $V : \mathbb{R}^n \to \mathbb{C}^n$ is a given (complex-valued) vector field. The equation is of some interest of its own, as it is a PDE analogue of the ODE

$$\tfrac{d}{dx}u(x) = u(x)a(x), \qquad x \in \mathbb{R}, \tag{$*_1$}$$

which is familiar to every first-year student of mathematics. But the author's interest in it comes from a study of a family of nonlinear and singular Schrödinger equations proposed by H.-D. Doebner and G. Goldin [2–4] as a model for nonlinear quantum mechanics (*Doebner–Goldin equations*). The role that Eq. ($*_n$) plays in solving these equations can be illustrated by looking at the equation

$$i\partial_t u + \tfrac{1}{2}\Delta u = {}^{i}D\left(\Delta u + \frac{|\nabla u|^2}{|u|^2}u\right) \tag{DGE}$$

0022-247X/01 $35.00

($D \in \mathbb{R}$) which is a comparatively simple member of the family. (For details, see [9, 13, 14].) On the right hand side of (DGE) the singular term $|\nabla u|^2/|u|^2$ appears which is not well-defined if the unknown function u has zeros. However, such zeros can be excluded by looking for solutions of the form $u = e^v$ only; it turns out that solutions of this form actually exist. Notice that $|\nabla u|^2/|u|^2 = |\nabla v|^2$. A second strategy consists in considering solutions u of the system

$$i\partial_t u + \tfrac{1}{2}\Delta u = {}^iD(\Delta u + |V|^2 u), \qquad \nabla u = uV, \tag{DGS}$$

as solutions of the original problem. Here no singularity appears; solutions u with zeros are not excluded as they are by the ansatz e^v; and solutions to (DGE) constructed by means of this ansatz are obviously solutions to (DGS).[1] Hence the new solution concept is more general than the first one, but the question arises as to whether it is "strictly more general," i.e., whether there are solutions to (DGS) which are not exponentials of certain functions v. In the case that V is the gradient of some function v the answer to this question is negative. Indeed, for every non-trivial solution u of the equation $\nabla u = u(\nabla v)$ there is a constant $c \in \mathbb{C}^*$ such that $u = ce^v$, which can be shown as in the one-dimensional case by differentiating the function ue^{-v}. So what it comes down to is the question of whether there are nontrivial solutions of $(*_n)$ if V is *not* the gradient of some function v. One aim of this paper is to show that the answer to this question is negative.

Of course, the last statement is meaningless as long as no definition of "solution" has been given; such a definition, in turn, depends heavily on the regularity of the vector field V. The first task of this paper will therefore be to devise a solution concept that is as general w.r.t. the class of admissible vector fields as possible. For instance, since $(*_1)$ can be solved uniquely if the function a is locally integrable, it seems natural to include locally integrable vector fields in the general theory as well.[2]

We now describe briefly the situation when V is continuous and classical analysis can be applied. This will illustrate two ideas which, after adjusting them to the "non-classical" situation of this paper, will be used later on.

THEOREM 1.1. *Let V be continuous and u a non-trivial (classically differentiable) solution of $(*_n)$. Then there exists a unique C^1-function v such that $\nabla v = V$ and $v(0) = 0$, and u is given by $u = u(0)e^v$.*

[1] This follows from the obvious fact that the function $u = e^v$ is a solution of the equation $\nabla u = u(\nabla v)$. Thus a solution u to (DGE) that can be written as $u = e^v$ solves (DGS) with $V = \nabla v$.

[2] Actually, the condition $V \in L^n_{loc}(\mathbb{R}^n)$ would be equally natural. It will be considered in connection with the weak solution concept below.

In the one-dimensional case we have the explicit expression $v(x) = \int_0^x a(y)dy$; in the general case the existence of v is a consequence of the well-known lifting theorem of classical analysis, cf., e.g., [11, Sect. III.6].[3] One first shows that u has no zeros. Then the lifting theorem implies that, for every $z \in \exp^{-1}(\{1\})$, there exists a unique continuously differentiable[4] function $v : \mathbb{R}^n \to \mathbb{C}$ such that $u = e^v$ and $v(0) = z$. Evidently, every such "C^1-lift" (or "logarithm") v of u solves the equation $\nabla v = V$. The proof of the fact that u has no zeros can be reduced to a one-dimensional problem: The function $y : \mathbb{R} \to \mathbb{C}$, $t \mapsto u(tx)$, is a solution of the ODE $y'(t) = y(t)\langle V(tx), x\rangle =: y(t)a(t)$. Hence $y(t) = u(0)e^{\int_0^t a(s)ds}$. Since u is not identically zero, we may w.l.o.g. assume that $u(0) \neq 0$, which implies $u(x) = y(1) = u(0)e^{\int_0^1 a(s)ds} \neq 0$. To sum up, the main ingredients of the proof are the reduction to the one-dimensional case and the application of the lifting theorem (existence of logarithms).

Let us mention that another proof can be given if V is differentiable. As before, one first shows that $u(x) \neq 0$ for all $x \in \mathbb{R}^n$. But now, in order to show that V is the gradient of of some function v, one simply verifies the integrability condition $\operatorname{curl} V := (\partial_j V_k - \partial_k V_j)_{j,\,k=1,\dots,n} = 0$

$$\begin{aligned} 0 &= (\operatorname{curl} \nabla u)_{jk} \overset{(*_n)}{=} (\operatorname{curl}(uV))_{jk} = (\partial_j u)V_k - (\partial_k u)V_j + u(\operatorname{curl} V)_{jk} \\ &\overset{(*_n)}{=} u(V_jV_k - V_kV_j) + u(\operatorname{curl} V)_{jk} = u(\operatorname{curl} V)_{jk} \end{aligned}$$

and applies Poincaré's lemma.

The paper is organized as follows. Our solution concept for $(*_n)$ is based on the function class $AC(\mathbb{R}^n)$, which is introduced in Section 2. In three respects this class seems to be a natural choice: First, it accommodates locally integrable vector fields; second, it generalizes the intuitive concept of weak solutions (see Definition 4.1) in the sense that weakly differentiable functions (see Definition 2.2) belong to $AC(\mathbb{R}^n)$ (Lemma 2.1) and weak solutions are in particular AC-solutions; and finally, by its very definition, which involves restricting of functions to parallels to the coordinate axes (see Definition 2.1), it incorporates the idea of going back two a one-dimensional situation. In Section 2 the technical properties of AC-functions needed later are proved, most importantly a lifting theorem, analogous to the one of classical analysis (Theorem 2.1). In Section 3 the class $\mathcal{V}$ of admissible vector fields (which contains the class of locally integrable vector fields) is introduced and AC-solutions are defined. The main result of

[3]Of course, $\mathbb{R}^n$ is simply connected. In fact, in the whole paper $\mathbb{R}^n$ can be replaced by an arbitrary simply connected subdomain of $\mathbb{R}^n$.

[4]Notice that, since the function uV is continuous, the solution u is in fact *continuously* differentiable.

this section (Theorem 3.1) states that the assertions of Theorem 1.1 are valid for non-trivial AC-solutions u as well if the vector field V belongs to the class $\mathscr{V}$. In Section 4 we discuss the weak solution concept. Simple examples show that there are in general no non-trivial weak solutions $(*_n)$ if the vector field V is only assumed to be locally integrable. However, a satisfactory existence and uniqueness theory can be established under the assumption $V \in L^n_{loc}(\mathbb{R}^n)$, which provides another natural generalization of the condition $a \in L^1_{loc}(\mathbb{R})$ of the one-dimensional case. Under this assumption on V we give an independent proof of the analogue of Theorem 1.1 (Theorem 4.1), mimicking the proof for the case of differentiable vector fields given above. Since the argument makes use of the Poincaré lemma, we need to prove a distributional version of it. This is done in Section 5 (Proposition 5.2) where the equation $\nabla v = V$ is studied. For the sake of completeness, we interpret this equation not only in the framework of distribution theory but also in the realm of AC-functions (Proposition 5.1). In the last section we discuss an aspect of Eq. $(*_n)$ which is more closely related to its role in the solution procedure of the Doebner–Goldin equations, namely the existence of non-trivial solutions under additional integrability conditions. Due to the quantum mechanical background of these equations, one has to insist that the solutions are square integrable. Moreover, the vector field V is constructed by solving a certain nonlinear evolution system. In order to make this evolution system tractable, one wants the vector field V to belong to $L^p(\mathbb{R}^n)$ for some $p \in [1, \infty]$. It turns out that no non-trivial L^2-solutions exist if $p \leq n$ (Propositions 6.1 and 6.2). To prove this some harmonic analysis is employed.

2. THE CLASS $AC(\mathbb{R}^N)$

DEFINITION 2.1. Let $j \in \{1, \ldots, n\}$. The set $AC_j = AC_j(\mathbb{R}^n)$ consists of all functions $\tilde{u} : \mathbb{R}^n \to \mathbb{C}$ that can be restricted to "almost every" parallel to the jth coordinate axis such that the restriction is an absolutely continuous (a.c.) function $\mathbb{R} \to \mathbb{C}$.[5] Here "almost every" is understood in the sense of the $(n-1)$-dimensional Lebesgue measure. More precisely, $\tilde{u} \in AC_j$ iff

$$\begin{pmatrix} \exists_{\widehat{N} \subset \mathbb{R}^{n-1} \text{ nullset}} \ \forall_{(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) \in \mathbb{R}^{n-1} \setminus \widehat{N}} \\ t \mapsto \tilde{u}(x_1, \ldots, x_{j-1}, t, x_{j+1}, \ldots, x_n) \text{ absolutely continuous } \mathbb{R} \to \mathbb{C} \end{pmatrix}.$$

We define

$$AC := AC(\mathbb{R}^n) := \{u : \mathbb{R}^n \to \mathbb{C} \mid \forall_{j \in \{1, \ldots, n\}} \exists_{\tilde{u}_j \in AC_j} \quad u \overset{\mu_n}{=} \tilde{u}_j\},$$

[5]A function $f : \mathbb{R} \to \mathbb{C}$ is called absolutely continuous if it is absolutely continuous on every compact subinterval of $\mathbb{R}$.

where $\stackrel{\mu_n}{=}$ denotes equality up to nullsets w.r.t. the n-dimensional Lebesgue measure μ_n.

Notation. Subsequently the following abbreviation will be convenient. Let $j, k \in \{1, \dots, n\}$, $j < k$, and $(x_1, \dots, x_n) \in \mathbb{R}^n$. Then components of $(x_1, \dots, x_n)$ marked with hats are to be left out and $\hat{x}_{(j)} \in \mathbb{R}^{n-1}$, $\hat{x}_{(j,k)} \in \mathbb{R}^{n-2}$ are defined by $\hat{x}_{(j)} := (x_1, \dots, \hat{x}_j, \dots, x_n)$ and $\hat{x}_{(j,k)} := (x_1, \dots, \hat{x}_j, \dots, \hat{x}_k, \dots, x_n)$.

Remark 2.1. (a) It is well known that a function $f : \mathbb{R} \to \mathbb{C}$ is absolutely continuous iff it is differentiable almost everywhere in $\mathbb{R}$ and its derivative is locally integrable. Thus, for an AC_j-function $\tilde{u}$, the jth partial derivative $\partial_j \tilde{u}$ is defined a.e. in $\mathbb{R}^n$ by

$$\begin{aligned}(\partial_j \tilde{u})(x_1, \dots, x_{j-1}, t, x_{j+1}, \dots, x_n)\\ := \tfrac{d}{ds}\big|_t\ \tilde{u}(x_1, \dots, x_{j-1}, s, x_{j+1}, \dots, x_n)\end{aligned}$$

for all $(x_1, \dots, \hat{x}_j, \dots, x_n) \in \mathbb{R}^{n-1} \setminus \widehat{N}$ and almost every $t \in \mathbb{R}$. Hence, for AC-functions u, all partial derivatives $\partial_j u$ $(j = 1, \dots, n)$ can be defined by $\partial_j u := \partial_j \tilde{u}_j$. (It is obvious that $\partial_j u$ is well-defined up to μ_n-nullsets.)

(b) Evidently the product and chain rules are valid for these partial derivatives.

DEFINITION 2.2. A function u is called *weakly differentiable* if it is locally integrable and all its partial derivatives in the sense of distributions are locally integrable as well.

For the class of weakly differentiable functions the following properties are well known (cf. [8, 5.2]).

LEMMA 2.1. (i) *Every weakly differentiable function belongs to* AC $(\mathbb{R}^n)$.

(ii) *A function* $u \in AC(\mathbb{R}^n)$, *which is locally integrable, is weakly differentiable*.

Moreover, the partial derivatives in the sense of distributions and in the sense of the space $AC(\mathbb{R}^n)$ *coincide almost everywhere*.

Remark 2.2. *A special version of Fubini's Theorem.* Many of the arguments given below can be dubbed "avoiding malign nullsets." For those arguments the following observation is indispensable. Let $N \subset \mathbb{R}^n$ be a nullset and $j \in \{1, \dots, n-1\}$. Then there exists a nullset $M \subset \mathbb{R}^j$ such that

$$\forall_{x \in \mathbb{R}^j \setminus M} \quad \mu_{n-j}\big(\{y \in \mathbb{R}^{n-j} \mid (x, y) \in N\}\big) = 0.$$

Remark 2.3. *Restriction to hyperplanes.* Since many of the following proofs involve induction over the space dimension, we convince ourselves once and for all that AC-functions can safely be restricted to lower-dimensional subspaces; for our purposes it will in fact be sufficient to consider subspaces with codimensions one and two. Let $u \in AC(\mathbb{R}^n)$. We are going to show that there is a nullset $M \subset \mathbb{R}^n$ such that for every $a \in \mathbb{R}^n \backslash M$, the functions $u^{(k)} : \mathbb{R}^{n-1} \to \mathbb{C}$ and $u^{(\ell,m)} : \mathbb{R}^{n-2} \to \mathbb{C}$, given by $u^{(k)}(\hat{x}_{(k)}) := u(x_1, \dots, a^k, \dots, x_n)$ and $u^{(\ell,m)}(\hat{x}_{(\ell,m)}) := u(x_1, \dots, a^\ell, \dots, a^m, \dots, x_n)$, are well defined (up to μ_{n-1}-nullsets and μ_{n-2}-nullsets, respectively) and belong to $AC(\mathbb{R}^{n-1})$ and $AC(\mathbb{R}^{n-2})$, respectively. To this end let $\tilde{u}_j \in AC_j(\mathbb{R}^n)$ ($j \in \{1, \dots, n\}$) be a family of functions defining u and $N \subset \mathbb{R}^n$, $\widehat{N} \subset \mathbb{R}^{n-1}$ nullsets such that $\tilde{u}_j|_{\mathbb{R}^n \backslash N} = u|_{\mathbb{R}^n \backslash N}$ and

$$\big(t \mapsto \tilde{u}_j(x_1, \dots, x_{j-1}, t, x_{j+1}, \dots, x_n)\big) \quad \text{is absolutely continuous}$$

for every $j \in \{1, \dots, n\}$ and $(x_1, \dots, \hat{x}_j, \dots, x_n) \in \mathbb{R}^{n-1} \backslash \widehat{N}$. For $a = (a^1, \dots, a^n) \in \mathbb{R}^n$ we define the sets $N_{a^k} \subset \mathbb{R}^{n-1}$, $N_{(a^\ell, a^m)} \subset \mathbb{R}^{n-2}$, $\widehat{N}_{a^k} \subset \mathbb{R}^{n-2}$ and $\widehat{N}_{(a^\ell, a^m)} \subset \mathbb{R}^{n-3}$ by

$$\begin{aligned}
N_{a^k} &:= \{\hat{x}_{(k)} \mid (x_1, \dots, a^k, \dots, x_n) \in N\} \\
N_{(a^\ell, a^m)} &:= \{\hat{x}_{(\ell,m)} \mid (x_1, \dots, a^\ell, \dots, a^m, \dots, x_n) \in N\} \\
\widehat{N}_{i, a^k} &:= \{\hat{x}_{(i,k)} \mid (x_1, \dots, \hat{x}_i, \dots, a^k, \dots, x_n) \in \widehat{N}\} \\
\widehat{N}_{j, (a^\ell, a^m)} &:= \{\hat{x}_{(j,\ell,m)} \mid (x_1, \dots, \hat{x}_j, \dots, a^\ell, \dots, a^m, \dots, x_n) \in \widehat{N}\} \\
\widehat{N}_{a^k} &:= \bigcup_{i \in \{1,\dots,n\} \backslash \{k\}} \widehat{N}_{i, a^k}, \qquad \widehat{N}_{(a^\ell, a^m)} := \bigcup_{j \in \{1,\dots,n\} \backslash \{\ell, m\}} \widehat{N}_{j, (a^k, a^\ell)},
\end{aligned}$$

respectively. Then there exist nullsets $I_k \subset \mathbb{R}$ and $I_{(\ell,m)} \subset \mathbb{R}^2$ such that

$$\begin{aligned}
\forall_{a^k \in \mathbb{R} \backslash I_k, (a^\ell, a^m) \in \mathbb{R}^2 \backslash I_{(\ell,m)}} \qquad & \mu_{n-1}(N_{a^k}) = \mu_{n-2}(N_{(a^\ell, a^m)}) = 0 \\
& \mu_{n-2}(\widehat{N}_{a^k}) = \mu_{n-3}(\widehat{N}_{(a^\ell, a^m)}) = 0
\end{aligned}$$

(by utilizing Remark 2.2 and taking finite unions of nullsets where necessary). Finally, define N_k and $N_{(k,\ell)} \subset \mathbb{R}^n$ by

$$\begin{aligned}
N_k &:= \{(x_1, \dots, x_{k-1}, t, x_{k+1}, \dots, x_n) \mid t \in I_k, \ \hat{x}_{(k)} \in \mathbb{R}^{n-1}\} \\
N_{(k,\ell)} &:= \{(x_1, \dots, x_{k-1}, s, x_{k+1}, \dots, x_{\ell-1}, t, x_{\ell+1}, \dots, x_n) \mid \\
&\qquad (s,t) \in I_{(k,\ell)}, \ \hat{x}_{(k,\ell)} \in \mathbb{R}^{n-2}\},
\end{aligned}$$

respectively; these sets are obviously nullsets. Now set

$$M := \bigcup_{1 \le k \le n} N_k \cup \bigcup_{1 \le \ell < m \le n} N_{(\ell,m)}$$

and let $a \in \mathbb{R}^n \backslash M$, $k, l, m \in \{1, \dots, n\}$, $\ell < m$. Notice that all the sets N_{a^k}, $N_{(a^\ell,\, a^m)}$, $\widehat{N}_{a^k}$, $\widehat{N}_{(a^\ell,\, a^m)}$ are nullsets. Now, for every $\hat{x}_{(k)} \in \mathbb{R}^{n-1} \backslash N_{a^k}$ and $\hat{y}_{(\ell,\, m)} \in \mathbb{R}^{n-2} \backslash N_{(a^\ell,\, a^m)}$, we have

$$(x_1, \dots, a^k, \dots, x_n),\ (y_1, \dots, a^\ell, \dots, a^m, \dots, y_n) \in \mathbb{R}^n \backslash N.$$

Hence $\tilde{u}_i^{(k)}(\hat{x}_{(k)}) := \tilde{u}_i(x_1, \dots, a^k, \dots, x_n) = u^{(k)}(\hat{x}_{(k)})$ and $\tilde{u}_j^{(\ell,m)}(\hat{y}_{(\ell,m)}) := \tilde{u}_j(y_1, \dots, a^\ell, \dots, a^m, \dots, y_n) = u^{(\ell,m)}(\hat{y}_{(\ell,\, m)})$ for all $i, j \in \{1, \dots, n\}$, $i \neq k$, $j \neq \ell, m$. Likewise, given $\hat{x}_{(i,\, k)} \in \mathbb{R}^{n-2} \backslash \widehat{N}_{a^k}$ and $\hat{y}_{(j,\, \ell,\, m)} \in \mathbb{R}^{n-3} \backslash \widehat{N}_{(a^\ell,\, a^m)}$, we get

$$\begin{aligned}&(x_1, \dots, \hat{x}_i, \dots, a^k, \dots, x_n),\\ &\quad (y_1, \dots, \hat{y}_j, \dots, a^\ell, \dots, a^m, \dots, y_n) \in \mathbb{R}^{n-1} \backslash \widehat{N},\end{aligned}$$

which implies that the functions $t \mapsto \tilde{u}_i^{(k)}(x_1, \dots, t, \dots, \hat{x}_k, \dots, x_n)$ and $t \mapsto \tilde{u}_j^{(\ell,\, m)}(y_1, \dots, t, \dots, \hat{y}_\ell, \dots, \hat{y}_m, \dots, y_n)$ are absolutely continuous. Thus $u^{(k)} \in AC(\mathbb{R}^{n-1})$ and $u^{(\ell,\, m)} \in AC(\mathbb{R}^{n-2})$.

In this sense the given AC-function u can be restricted to all the hyperplanes $\{(x_1, \dots, a^k, \dots, x_n) \mid \hat{x}_{(k)} \in \mathbb{R}^{n-1}\}$ and $\{(\dots, a^\ell, \dots, a^m, \dots) \mid \hat{x}_{(\ell,\, m)} \in \mathbb{R}^{n-2}\}$ provided a avoids the μ_n-nullset M. Moreover, these restrictions and differentiation “commute”; i.e., $\partial_i u^{(k)} \overset{\mu_{n-1}}{=} (\partial_i u)^{(k)}$ and $\partial_j u^{(\ell,m)} \overset{\mu_{n-2}}{=} (\partial_j u)^{(\ell,m)}$, which is now obvious from the definition of the partial derivatives of AC-functions.

LEMMA 2.2. *Let $u \in AC$ be a function with $\nabla u \overset{\mu_n}{=} 0$. Then there exists a constant $c \in \mathbb{C}$ such that $u \overset{\mu_n}{=} c$.*

Proof. Induction over the space dimension n. $n = 1$ is clear.

$n - 1 \to n$. Let $\tilde{u}_n \in AC_n(\mathbb{R}^n)$ and $N \subset \mathbb{R}^n$, $\widehat{N} \subset \mathbb{R}^{n-1}$ be nullsets such that $u|_{\mathbb{R}^n \backslash N} = \tilde{u}_n|_{\mathbb{R}^n \backslash N}$ and $y := (t \mapsto \tilde{u}_n(x_1, \dots, x_{n-1}, t))$ is absolutely continuous for every $(x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1} \backslash \widehat{N}$. Choose $a^n \in \mathbb{R}$ such that u can be restricted (in the sense of Remark 2.3) to the hyperplane $\{(x_1, \dots, x_{n-1}, a^n) \mid (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}\}$ (in particular $\mu_{n-1}(N_{a^n}) = 0$). From Remark 2.3 we know that $\hat{\nabla}^n u^{(n)} \overset{\mu_{n-1}}{=} 0$, where $\hat{\nabla}^n := (\partial_1, \dots \partial_{n-1})$. Hence, according to the assumption of the induction, there is a constant $c \in \mathbb{C}$ and a nullset $\widehat{M} \subset \mathbb{R}^{n-1}$ such that $u^{(n)}|_{\mathbb{R}^{n-1} \backslash \widehat{M}} = c$. From $y' \overset{\mu_1}{=} 0$ it follows that

$$\begin{aligned}\tilde{u}_n(x_1, \dots, x_{n-1}, t) &= \tilde{u}_n(x_1, \dots, x_{n-1}, a^n) = u(x_1, \dots, x_{n-1}, a^n)\\ &= u^{(n)}(x_1, \dots, x_{n-1}) = c\end{aligned}$$

for all $(x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1} \backslash (\widehat{N} \cup \widehat{M} \cup N_{a^n})$ and $t \in \mathbb{R}$. ∎

PROPOSITION 2.1. (Lifting of AC-Functions). *Let $u \in AC$ and assume that there are functions $\tilde{u}_j \in AC_j$ $(j \in \{1, \ldots, n\})$ and nullsets $N \subset \mathbb{R}^n$ and $\widehat{N} \subset \mathbb{R}^{n-1}$ such that $u|_{\mathbb{R}^n \setminus N} = \tilde{u}_j|_{\mathbb{R}^n \setminus N}$,*

$$y := (t \mapsto \tilde{u}_j(x_1, \ldots, x_{j-1}, t, x_{j+1}, \ldots, x_n)) \text{ a.c.}$$
$$\text{and} \quad \forall_{t \in \mathbb{R}} \qquad y(t) \neq 0$$

for every $j \in \{1, \ldots, n\}$ and $(x_1, \ldots, \hat{x}_j, \ldots, x_n) \in \mathbb{R}^{n-1} \setminus \widehat{N}$. Then there exists an "AC-lift" of u, i.e., a function $v \in AC$ such that $u = e^v$. Moreover, for any two AC-lifts $v_1, v_2 \in AC$ of u, there exists a unique $c \in 2\pi i\mathbb{Z}$ such that $v_1 - v_2 \overset{\mu_n}{=} c$.

Proof. *Uniqueness*. Let $v_1, v_2 \in AC$ be two AC-lifts of u. Then the chain rule gives $0 = \nabla(u - u) = \nabla(e^{v_1} - e^{v_2}) = u\nabla(v_1 - v_2)$, which implies $\nabla(v_1 - v_2) = 0$ and therefore (cf. Lemma 2.2) $v_1 - v_2 \overset{\mu_n}{=} c$.

Existence. Induction over the space dimension n. The assumption of this induction reads,

$$\begin{aligned} &\text{For every } u \in AC(\mathbb{R}^n) \text{ satisfying the assumption of the} \\ &\quad \text{proposition there exist nullsets } M^{(n)} \subset \mathbb{R}^n \text{ and } \widehat{M}^{(n)} \subset \mathbb{R}^{n-1} \\ &\quad \text{as well as functions } \tilde{v}_1 \in AC_1(\mathbb{R}^n), \ldots, \tilde{v}_n \in AC_n(\mathbb{R}^n) \\ &\quad \text{such that } \tilde{u}_i|_{G^{(n)}} = e^{\tilde{v}_i}|_{G^{(n)}} \text{ and } \tilde{v}_i|_{G^{(n)} \setminus M^{(n)}} = \tilde{v}_j|_{G^{(n)} \setminus M^{(n)}} \\ &\quad \text{for all } i, j \in \{1, \ldots, n\}, i < j. \text{ Here } G^{(n)} \text{ denotes the} \\ &\quad (\widehat{M}^{(n)}\text{-dependent) set } G^{(n)} := \{(x_1, \ldots, x_n) \mid \\ &\quad \forall_{j \in \{1, \ldots, n\}}\ \hat{x}_{(j)} \notin \widehat{M}^{(n)}\}. \end{aligned} \tag{A$_n$}$$

In the case $n = 1$ the well-known lifting theorem of classical analysis (mentioned in the Introduction) can be applied directly.

Preliminary Remark Regarding the Proof of the Case $n = 2$. Here the lifting theorem is used in the following way: (1) The functions $\tilde{u}_j$ $(j = 1, 2)$ restricted to parallels to the jth coordinate axis define a.c. curves which can be lifted uniquely to a.c. curves if the initial values are prescribed. These lifted curves patched together define the functions $\tilde{v}_j$. (The "correct" initial data are given by the curves d_2 and d_1 below.)

(2) To show that the functions $\tilde{v}_1$ and $\tilde{v}_2$ actually coincide a.e., the uniqueness part of the lifting theorem is evoked: Let $\tilde{c}_1, \tilde{c}_2 : [0, 1] \to \mathbb{C}^*$ be continuous curves satisfying $\tilde{c}_1(0) = \tilde{c}_2(0)$ and $\tilde{c}_1(1) = \tilde{c}_2(1)$ and $\tilde{d}_1, \tilde{d}_2 : [0, 1] \to \mathbb{C}$ continuous lifts. Then $\tilde{d}_1(0) = \tilde{d}_2(0)$ implies $\tilde{d}_1(1) = \tilde{d}_2(1)$.

Proof of the Case $n = 2$. We choose $M^{(2)}$ and $\widehat{M}^{(2)}$ to be N and $\widehat{N}$, respectively. Let $a = (a^1, a^2) \in G^{(2)} \setminus N$ and $z \in \exp^{-1}(\{u(a)\})$. Let $d_1 : \mathbb{R} \to \mathbb{C}$ be the unique absolutely continuous lift of $\tilde{u}_1^2 = (t \mapsto \tilde{u}_1(t, a^2))$ such that $d_1(a^1) = z$ and $d_2 : \mathbb{R} \to \mathbb{C}$ be the unique a.c. lift of $\tilde{u}_2^1 = (t \mapsto \tilde{u}_2(a^1, t))$ s.t. $d_2(a^2) = z$. Define $\tilde{v}_1, \tilde{v}_2 : G^{(2)} \to \mathbb{C}$ by

$$\forall_{x_2 \in \mathbb{R}\setminus\widehat{N}} \quad \big(t \mapsto \tilde{v}_1(t, x_2)\big) := \begin{pmatrix} \text{unique a.c. lift of } (t \mapsto \tilde{u}_1(t, x_2)) \\ \text{s.t. } \tilde{v}_1(a^1, x_2) = d_2(x_2) \end{pmatrix}$$

and

$$\forall_{x_1 \in \mathbb{R}\setminus\widehat{N}} \quad \big(t \mapsto \tilde{v}_2(x_1, t)\big) := \begin{pmatrix} \text{unique a.c. lift of } (t \mapsto \tilde{u}_2(x_1, t)) \\ \text{s.t. } \tilde{v}_2(x_1, a^2) = d_1(x_1) \end{pmatrix};$$

it is easy to see that $\tilde{v}_j \in AC_j(\mathbb{R}^2)$ $(j = 1, 2)$. Therefore the proof of the assertion

$$\tilde{v}_1|_{G^{(2)}\setminus N} = \tilde{v}_2|_{G^{(2)}\setminus N} \tag{1}$$

will complete the proof of the case $n = 2$. We start with some notation: given two curves $c_j : [\,\alpha_j, \beta_j\,] \to \mathbb{R}^2$ $(j = 1, 2)$ we define their "concatenation" $c_1 * c_2 : [\,0, 1\,] \to \mathbb{R}^2$ by

$$(c_1 * c_2)(t) := \begin{cases} c_1(2(\beta_1 - \alpha_1)t + \alpha_1), & t \in [\,0, \frac{1}{2}\,] \\ c_2(2(\beta_2 - \alpha_2)t + 2\alpha_2 - \beta_2), & t \in [\,\frac{1}{2}, 1\,]. \end{cases}$$

The curve $c_1 * c_2$ is obviously continuous if c_1 and c_2 are continuous and $c_1(\beta_1) = c_2(\alpha_2)$. Let $(x_1, x_2) \in G^{(2)} \setminus N$; for convenience we assume $x_1 \geq a^1$ and $x_2 \geq a^2$ (otherwise certain curves simply have to be made to run "backwards"). Consider the continuous curves $\tilde{c}_j, \tilde{d}_j : [\,0, 1\,] \to \mathbb{C}$, defined by $\tilde{c}_1 := (\tilde{u}_1(., a^2)|_{[\,a^1, x_1\,]}) * (\tilde{u}_2(x_1, .)|_{[\,a^2, x_2\,]})$, $\tilde{d}_1 := (d_1|_{[\,a^1, x_1\,]}) * (\tilde{v}_2(x_1, .)|_{[\,a^2, x_2\,]})$, $\tilde{c}_2 := (\tilde{u}_2(a^1, .)|_{[\,a^2, x_2\,]}) * (\tilde{u}_1(., x_2)|_{[\,a^1, x_1\,]})$, and $\tilde{d}_2 := (d_2|_{[\,a^2, x_2\,]}) * (\tilde{v}_1(., x_2)|_{[\,a^1, x_1\,]})$. Evidently, $\tilde{d}_1$ and $\tilde{d}_2$ are continuous lifts of $\tilde{c}_1$ and $\tilde{c}_2$, respectively, which satisfy $\tilde{d}_1(0) = z = \tilde{d}_2(0)$. Now the classical lifting theorem implies, because of $\tilde{c}_1(0) = e^z = \tilde{c}_2(0)$ and $\tilde{c}_1(1) = \tilde{u}_2(x_1, x_2) = \tilde{u}_1(x_1, x_2) = \tilde{c}_2(1)$,

$$\tilde{v}_2(x_1, x_2) = \tilde{d}_1(1) = \tilde{d}_2(1) = \tilde{v}_1(x_1, x_2).$$

Hence, (1).

Since the proof of "$(\mathrm{A}_{n-1}) \Rightarrow (\mathrm{A}_n)$" requires no new ideas but rather some additional notation, it will be omitted here. ∎

3. SOLVING $(*_N)$ WITHIN THE CLASS AC

DEFINITION 3.1. (i) The set $\mathscr{V} = \mathscr{V}(\mathbb{R}^n)$ is defined as the set of all vector fields $V : \mathbb{R}^n \to \mathbb{C}^n$ such that

$$\begin{pmatrix} \exists_{\widehat{N} \subset \mathbb{R}^{n-1} \text{ nullset}} \ \forall_{j \in \{1,\dots,n\}, (x_1,\dots,\hat{x}_j,\dots,x_n) \in \mathbb{R}^{n-1}\setminus\widehat{N}} \\ t \mapsto V(x_1,\dots,x_{j-1},t,x_{j+1},\dots,x_n) \quad \text{locally integrable} \end{pmatrix}.$$

(ii) Let $V \in \mathscr{V}$. Then a function $u : \mathbb{R}^n \to \mathbb{C}$ is called a *solution* of $(*_n)$ if it belongs to the set AC and fulfills $(*_n)$ almost everywhere in $\mathbb{R}^n$.

(It is obvious that the pointwise defined product of an AC- and a $\mathscr{V}$-function belongs to $\mathscr{V}$.)

THEOREM 3.1. *Let $V : \mathbb{R}^n \to \mathbb{C}^n$ be a vector field which belongs to $\mathscr{V}$ and $u : \mathbb{R}^n \to \mathbb{C}$ a non-trivial solution of* $(*_n)$. *Then there exists an* (*up to additive constants* $\in 2\pi i\mathbb{Z}$ *unique*) *AC-function v such that $\nabla v = V$ and $u = e^v$.*

Proof. It is obvious that it suffices to show that u possesses an AC-lift v. To this end we first prove that

$$u(x) \neq 0 \qquad \text{for almost every } x \in \mathbb{R}^n. \tag{2}$$

Proof of (2) *by Induction over the Space Dimension n.* (A) The case $n = 1$ was mentioned in the Introduction. (B) $n - 1 \to n$. Define the functions $u^t : \mathbb{R}^{n-1} \to \mathbb{C}$ and $\widehat{V}^t : \mathbb{R}^{n-1} \to \mathbb{C}^{n-1}$ by $u^t(\hat{x}) := u(\hat{x}, t)$ and $\widehat{V}^t(\hat{x}) := (V_1(\hat{x}, t), \dots, V_{n-1}(\hat{x}, t))$, respectively. From Remark 2.3 it follows (after changing the notation slightly) that there is a μ_1-nullset $I_0 \subset \mathbb{R}$ such that, for every $t \in \mathbb{R}\setminus I_0$, the function u^t is a solution of $(*_{n-1})$ with V replaced by $\widehat{V}^t$; i.e., $u^t \in AC(\mathbb{R}^{n-1})$, $\widehat{V}^t \in \mathscr{V}(\mathbb{R}^{n-1})$, and $\widehat{\nabla} u^t \overset{\mu_{n-1}}{=} u^t \widehat{V}^t$, where $\widehat{\nabla} := (\partial_1, \dots, \partial_{n-1})$. Since $u^t(\hat{x})$ cannot be zero for almost every $\hat{x} \in \mathbb{R}^{n-1}$ and $t \in \mathbb{R}$ (as this would imply $u(x) = 0$ for almost every $x \in \mathbb{R}^n$), there is a $t \in \mathbb{R}\setminus I$ for every given nullset $I \subset \mathbb{R}$ such that u^t is non-trivial (in the sense of the $(n-1)$-dimensional Lebesgue measure). Hence, according to the assumption of the induction, $u^t(\hat{x}) \neq 0$ for almost every $\hat{x} \in \mathbb{R}^{n-1}$. Thus, there exist in particular $t_0 \in \mathbb{R}\setminus I_0$ and $\widehat{N}_0 \subset \mathbb{R}^{n-1}$ with $\mu_{n-1}(N_0) = 0$ such that

$$\forall_{\hat{x} \in \mathbb{R}^{n-1}\setminus\widehat{N}_0} \qquad u^{t_0}(\hat{x}) \neq 0. \tag{3}$$

Choose a function $\tilde{u}_n \in AC_n$ and nullsets $N \subset \mathbb{R}^n$, $\widehat{N} \subset \mathbb{R}^{n-1}$ such that $u|_{\mathbb{R}^n\setminus N} = \tilde{u}_n|_{\mathbb{R}^n\setminus N}$, $(\partial_n \tilde{u}_n)|_{\mathbb{R}^n\setminus N} = (\tilde{u}_n V)|_{\mathbb{R}^n\setminus N}$ and that, for every $\hat{x} \in \mathbb{R}^{n-1}\setminus\widehat{N}$, the set $I_{\hat{x}} := \{t \in \mathbb{R} \mid (\hat{x}, t) \in N\} \subset \mathbb{R}$ is a μ_1-nullset and the functions $y := (t \mapsto \tilde{u}_n(\hat{x}, t))$ and $a := (t \mapsto V(\hat{x}, t))$ are a.c. and locally integrable (l.i.), respectively. Let $\hat{x} \in \mathbb{R}^{n-1} \setminus \widehat{N}$. Then the a.c. function y is differentiable outside a certain μ_1-nullset $J_{\hat{x}} \subset \mathbb{R}$. Thus, we have

$$\forall_{t \in \mathbb{R} \setminus (I_{\hat{x}} \cup J_{\hat{x}})} \qquad y'(t) = (\partial_n \tilde{u}_n)(\hat{x}, t) = \tilde{u}_n(\hat{x}, t) V_n(\hat{x}, t) = y(t) a(t).$$

From (A) we get that either $y(t) \neq 0$ for all $t \in \mathbb{R}$, or $y(t) = 0$ for all $t \in \mathbb{R}$. Hence, thanks to (3), $\tilde{u}_n(\hat{x}, t) \neq 0$ for all $(\hat{x}, t) \in (\mathbb{R}^{n-1} \setminus (\widehat{N} \cup \widehat{N}_0)) \times \mathbb{R}$. This completes the proof of (2). The fact that Proposition 2.1 can be applied to show the existence of an AC-lift v of u follows from the subsequent

LEMMA. *Let $u \in AC$ be a solution of $(*_n)$ and $\tilde{u}_j \in AC_j$ $(j \in \{1, \ldots, n\})$ a family of functions which defines u. Then there exist nullsets $N \subset \mathbb{R}^n$ and $\widehat{N} \subset \mathbb{R}^{n-1}$ such the triple $(\{\tilde{u}_j\}_{j \in \{1,\ldots,n\}}, N, \widehat{N})$ fulfills the assumption of Proposition* 2.1.

Proof of the Lemma. Nullsets N and $\widehat{N}$ can be found such that $\tilde{u}_j(x) = u(x) \neq 0$ and $(\partial_j \tilde{u}_j)(x) = (\tilde{u}_j V_j)(x)$ for all $x \in \mathbb{R}^n \setminus N$ and

$$\begin{pmatrix} \mu_1(\{t \in \mathbb{R} \mid (x_1, \ldots, x_{j-1}, t, x_{j+1}, \ldots, x_n) \in N\}) = 0 \\ y := (t \mapsto \tilde{u}_j(\ldots, t, \ldots)) \text{ a.c.}, \quad a := (t \mapsto V_j(\ldots, t, \ldots)) \text{ l.i.} \end{pmatrix}$$

for all $\hat{x}_{(j)} \in \mathbb{R}^{n-1} \setminus \widehat{N}$. As in the proof of (2), the a.c. function y solves the ODE $y' \overset{\mu_1}{=} a \cdot y$; therefore, $\tilde{u}_j(x_1, \ldots, t, \ldots, x_n) = y(t) \neq 0$ for every $\hat{x}_{(j)} \in \mathbb{R}^{n-1} \setminus \widehat{N}$ and $t \in \mathbb{R}$. ∎

4. OTHER SOLUTION CONCEPTS

Probably the most natural solution is that of a weak solution.

DEFINITION 4.1. Let V be a locally integrable vector field. A locally integrable function u is called a *weak solution of* $(*_n)$ if

(1) the (pointwise) product uV of u and V is a locally integrable function

(2) equation $(*_n)$ is fulfilled in the sense of distributions; i.e.,

$$\forall_{\varphi \in \mathscr{D}(\mathbb{R}^n),\ j \in \{1, \ldots, n\}} \qquad -\int u\, \partial_j \varphi - \int u V_j\, \varphi = 0,$$

where $\mathscr{D}(\mathbb{R}^n)$ denotes the Fréchet space of all C^∞-functions with compact support.

(Of course the definition implies that every weak solution is actually weakly differentiable, thus $u \in W^{1,1}_{loc}(\mathbb{R}^n) \hookrightarrow L^{n/(n-1)}_{loc}(\mathbb{R}^n)$.)

LEMMA 4.1. (i) *Every weak solution is a solution in the sense of Definition* 2.1. (ii) *A solution $u \in AC$* (*in the sense of Definition* 2.1) *is a weak solution if $uV \in L^1_{loc}$.*

The trouble with this solution concept is that the existence of solutions cannot be guaranteed, i.e., vector fields $V \in L^p_{loc}(\mathbb{R}^n) \setminus L^n_{loc}(\mathbb{R}^n)$ can be found such that $(*_n)$ has no non-trivial weak solution. More precisely, for the non-trivial AC-solutions u (which exist as we know from the previous section) the product uV is in general not locally integrable. However, the condition $V \in L^n_{loc}(\mathbb{R}^n)$ is sufficient to guarantee the existence of non-trivial weak solutions.

LEMMA 4.2. *Let $V \in L^n_{loc}(R^n)$ and $v \in L^1_{loc}(\mathbb{R}^n)$ such that $\nabla v = V$. Then $e^v V \in L^1_{loc}(\mathbb{R}^n)$.*

Proof. From [16, Theorem 2.9.1], we get $e^v \in L^{n/(n-1)}_{loc}(\mathbb{R}^n)$, which implies the assertion by the Hölder inequality. ∎

THEOREM 4.1. *Let $V \in L^n_{loc}(\mathbb{R}^n)$ and $u \in L^1_{loc}(\mathbb{R}^n)$ be a non-trivial weak solution of $(*_n)$. Then there exists a* (*up to additive constants $\in 2\pi i\mathbb{Z}$ unique*) *weakly differentiable function v such that $\nabla v = V$ and $u = e^v$.*

Proof. We know from (2) that $u \neq 0$ almost everywhere. Set $E := \frac{u}{|u|}$ (almost everywhere in $\mathbb{R}^n$ well-defined), $R := Re(V)$, $S := Im(V)$. Then it is easy to see that the functions $|u|$ and E are weakly differentiable and satisfy

$$\nabla|u| - |u|R = 0 \qquad \text{and} \qquad \nabla E - iES = 0. \tag{4}$$

The crucial point is to show that

$$\operatorname{curl} V = 0. \tag{5}$$

Once this has been done, the assertion follows from the distributional Poincaré lemma 5.2.

Proof of (5). Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and $\langle .\, , . \rangle$ denote the pairing of $\mathcal{D}(\mathbb{R}^n)$ and $\mathcal{D}'(\mathbb{R}^n)$. Then

$$\begin{aligned}
-\langle \partial_j R_k\,, \varphi \rangle &= \int \frac{\partial_k |u|}{|u|}\, \partial_j \varphi = \lim_{\varepsilon\to 0} \int \frac{\partial_k(|u|+\varepsilon)}{|u|+\varepsilon}\, \partial_j\varphi \\
&= \lim_{\varepsilon\to 0} \int (\partial_k \ln(|u|+\varepsilon))(\partial_j\varphi) = -\lim_{\varepsilon\to 0}\int \ln(|u|+\varepsilon)\, \partial_k\partial_j\varphi \\
&= -\lim_{\varepsilon\to 0}\int \ln(|u|+\varepsilon)\, \partial_j\partial_k\varphi = -\lim_{\varepsilon\to 0}\int \frac{\partial_j(|u|+\varepsilon)}{|u|+\varepsilon}\, \partial_k\varphi \\
&= -\langle \partial_k R_j\,, \varphi \rangle.
\end{aligned}$$

Let $(E_\varepsilon := E * \eta_\varepsilon)_{\varepsilon>0} \subset C^\infty(\mathbb{R}^n)$ be a regularization of E; i.e.,

$$E_\varepsilon \xrightarrow{L^1_{loc}} E \quad \text{and} \quad \nabla E_\varepsilon = (\nabla E) * \eta_\varepsilon = (iES) * \eta_\varepsilon \xrightarrow{L^2_{loc}} iES \qquad (\varepsilon \to 0)$$

(because $ES \in L^2_{loc}$). Then

$$\begin{aligned}
-i\,\langle (\operatorname{curl} S)_{jk}\,, \varphi\,\rangle &= \int \left(\tfrac{\partial_j E}{E}\,\partial_k \varphi - \tfrac{\partial_k E}{E}\,\partial_j \varphi \right) \\
&= -\lim_{\varepsilon\to 0} \int \tfrac{1}{E}(\partial_j E_\varepsilon \partial_k \varphi - \partial_k E_\varepsilon \partial_j \varphi) \\
&= \lim_{\varepsilon\to 0} \Big\{ \int -\tfrac{1}{E^2}(\partial_k E \partial_j E_\varepsilon - \partial_j E \partial_k E_\varepsilon)\varphi + \cdots \\
&\qquad\qquad \cdots + \int \tfrac{1}{E}(\partial_k \partial_j E_\varepsilon - \partial_j \partial_k E_\varepsilon)\varphi \Big\} \\
&= \lim_{\varepsilon\to 0} \int -\tfrac{i}{E}(S_k \partial_j E_\varepsilon - S_j \partial_k E_\varepsilon)\varphi \\
&= \int (S_k S_j - S_j S_k)\varphi = 0.
\end{aligned}$$

This proves (5). ∎

Distributional Solutions. In devising other—for instance, distributional—solution concepts one has to make sure that the product uV is well defined, which is in general not the case if one of the factors is not a function. In two cases the product can be defined within the realm of distributions, namely if one of the factors is a smooth function.

(1) $V \in C^\infty$. In this case it can be shown along the lines of [7, Sect. 4.4], that every distributional solution $u \in \mathscr{D}'(\mathbb{R}^n)$ can be represented by a smooth function.

(2) One might be inclined to believe that the requirement $u \in C^\infty$ implies $V \in C^\infty$. That this is in fact not so can already be seen in the one-dimensional case. The C^∞-function $u = \frac{1}{2}|x|^2$ is a distributional solution of $(*_1)$, where v is given by the "proper" (non-regular) distribution p.v.$\frac{1}{x}$. It would be interesting to study Eq. $(*_n)$ in the realm of generalized functions (Colombeau algebras). A step in this direction can be found in [10].

5. TWO POINCARÉ LEMMAS

In this section we investigate the condition for solving $(*_n)$, namely the existence of a function v such that $\nabla v = V$. (One may say that this is the simplest PDE, cf. [5 2.4], and $(*_n)$ the "second simplest.") We reiterate the remark made in footnote 3 that in all the statements the domain $\mathbb{R}^n$ could be replaced by an arbitrary simply connected subdomain $\Omega \subset \mathbb{R}^n$.

PROPOSITION 5.1 (AC Poincaré Lemma). *Let $V \in (AC(\mathbb{R}^n))^n$ be a vector field with* $\operatorname{curl} V = 0$ (*this condition is to be understood in the sense of the proof below*). *Then there is a $v \in AC(\mathbb{R}^n)$ which solves the equation $\nabla v = V$ and any two solutions of this equation differ by a constant. Moreover, v is locally integrable if V is locally integrable.*

Proof. Let $\widetilde{V}_{j,k} \in AC_k(\mathbb{R}^n)$ ($k \in \{1, \ldots, n\}$) be a family of functions defining $V_j \in AC(\mathbb{R}^n)$ ($j \in \{1, \ldots, n\}$) and $N \subset \mathbb{R}^n$, $\widehat{N} \subset \mathbb{R}^{n-1}$ nullsets such that $\widetilde{V}_{j,k}|_{\mathbb{R}^n \setminus N} = \widetilde{V}_{j,\ell}|_{\mathbb{R}^n \setminus N}$, $\partial_j \widetilde{V}_{k,j}|_{\mathbb{R}^n \setminus N} = \partial_k \widetilde{V}_{j,k}|_{\mathbb{R}^n \setminus N}$ and

$$\left(\begin{array}{c} \mu_1\Big(\{t \in \mathbb{R} \mid (x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_n) \in N\}\Big) = 0 \\ (t \mapsto \widetilde{V}_{j,k}(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_n)) \text{ a.c.} \end{array} \right)$$

for all $j, k \in \{1, \ldots, n\}$, $\hat{x}_{(k)} \in \mathbb{R}^{n-1} \setminus \widehat{N}$. Choose an element $a \in \mathbb{R}^n$ and a nullset $M \subset \mathbb{R}^n$ such that $(a_1, \ldots, a_{j-1}, x_{j+1}, \ldots, x_n) \notin \widehat{N}$, $(a_1, \ldots, a_{k-1}, x_k, \ldots, x_n) \notin N$ and $\mu_1\big(\{t \mid (a_1, \ldots, a_{j-1}, t, x_{j+1}, \ldots, \hat{x}_k, \ldots, x_n) \in \widehat{N}\}\big) = 0$, $\mu_1\big(\{t \mid (a_1, \ldots, a_{j-1}, t, x_{j+1}, \ldots, x_n) \in N\}\big) = 0$ for all $x \in \mathbb{R}^n \setminus M$, $j, k \in \{1, \ldots, n\}$, $j < k$. The existence of an element a (in fact, almost every $a \in \mathbb{R}^n$ is suitable) and a nullset M with this property can be proved by a repeated application of Remark 2.2. Working out the details requires a fair amount of diligence and perseverance and is left to the intrepid reader. Now we are ready to define $\tilde{v}_k$ by

$$\begin{aligned} \tilde{v}_k(x) := \sum_{j=1}^{k} \int_{a_j}^{x_j} \widetilde{V}_{j,k}(a_1, \ldots, a_{j-1}, t, x_{j+1}, \ldots, x_n)dt + \cdots \\ + \sum_{j=k+1}^{n} \int_{a_j}^{x_j} \widetilde{V}_{j,j}(a_1, \ldots, a_{j-1}, t, x_{j+1}, \ldots, x_n)dt. \end{aligned}$$

It is a matter of simple calculations to show that $\tilde{v}_k$ and $\tilde{v}_\ell$ coincide almost everywhere and that $\partial_k \tilde{v}_k \overset{\mu_n}{=} \widetilde{V}_{k,1}$. ∎

PROPOSITION 5.2 (Distributional Poincaré Lemma). *Let V be a distributional vector field with* $\operatorname{curl} V = 0$. *Then there is a distribution $v \in \mathcal{D}'(\mathbb{R}^n)$ which solves the equation $\nabla v = V$ and any two solutions of this equation differ by a constant. Moreover, v is a weakly differentiable function if V is a regular distribution.*

It should be mentioned that in [1] a complete theory of differential forms whose coefficients are distributions, called "currents," is developed which could be applied here. However, our objective is to give an elementary proof of the result.

Proof. The proof is an adaption of the proof given in [5, Theorem 2.4.1], for the one-dimensional case. Let $\varphi_j \in \mathcal{D}(\mathbb{R})$ with $\int_{\mathbb{R}} \varphi_j = 1$ and define the operators

$$\nu_j\varphi := \left(\int_{\mathbb{R}^j} \varphi(x_{(j)}, x^{(j+1)})dx_{(j)}\right) \varphi_1(x_1)\cdots\varphi_j(x_j), \qquad \nu_0\varphi := \varphi$$

$$I_j\varphi := \int_{-\infty}^{x_j} \left\{\varphi(x_{(j-1)}, t, x^{(j+1)}) - \left(\int_{\mathbb{R}} \varphi(x_{(j-1)}, \tau, x^{(j+1)})d\tau\right) \varphi_j(t)\right\}dt$$

$$\mu_j\varphi := I_j(\nu_{j-1}\varphi) = \int_{-\infty}^{x_j} (\nu_{j-1}\varphi - \nu_j\varphi)(x_{(j-1)}, t, x^{(j+1)})dt,$$

where $x_{(\ell)} := (x_1, \ldots, x_\ell)$ and $x^{(m)} := (x_m, \ldots, x_n)$ denote the first ℓ and last $n-m$ components of $x = (x_1, \ldots, x_n)$, respectively. The operators ν, I are obviously continuous $\mathcal{D}(\mathbb{R}^n) \to \mathcal{D}(\mathbb{R}^n)$ and we have

$$\forall_{j,\,k\,\in\{1,\ldots,n\},\ j<k} \qquad \mu_k(\partial_j\varphi) = I_k(\nu_{k-1}\partial_j\varphi) = 0.$$

(it is obvious from the definition of the operator ν_{k-1} that $\nu_{k-1}(\partial_j\varphi) = 0$ for every $j \le k-1$)

$$\begin{aligned}\forall_{j,\,k\,\in\{1,\ldots,n\},\ j>k} \qquad \mu_k(\partial_j\varphi) &= I_k(\nu_{k-1}\partial_j\varphi) = I_k(\partial_j\nu_{k-1}\varphi)\\ &= \partial_j I_k(\nu_{k-1}\varphi) = \partial_j\mu_k\varphi\end{aligned}$$

$$\mu_j(\partial_j\varphi) = I_j(\nu_{j-1}\partial_j\varphi) = I_j(\partial_j\nu_{j-1}\varphi) = \nu_{j-1}\varphi.$$

$$\partial_j\mu_j(\varphi) = \nu_{j-1}\varphi - \left(\int_{\mathbb{R}}(\nu_{j-1}\varphi)(x_{(j-1)}, \tau, x^{(j+1)})d\tau\right)\varphi_j(x_j) = \nu_{j-1}\varphi - \nu_j\varphi.$$

Now we define $v \in \mathcal{D}'(\mathbb{R}^n)$ by $-\langle v\,, \varphi\,\rangle := \sum_{k=1}^n \langle V_k\,, \mu_k\varphi\,\rangle$ and calculate

$$\begin{aligned}\langle v\,, \partial_j\varphi\,\rangle &= \sum_{k=1}^{j-1} \langle V_k\,, \underbrace{\mu_k(\partial_j\varphi)}_{\partial_j\mu_k\varphi}\,\rangle + \langle V_j\,, \underbrace{\mu_j(\partial_j\varphi)}_{=\nu_{j-1}\varphi}\,\rangle + \sum_{k=j+1}^{n} \langle V_k\,, \underbrace{\mu_k(\partial_j\varphi)}_{=0}\,\rangle\\ &= \sum_{k=1}^{j-1} \langle V_j\,, \underbrace{\partial_k\mu_k\varphi}_{=\nu_{k-1}\varphi-\nu_k\varphi}\,\rangle + \nu_{j-1}\varphi = \langle V_j\,, \nu_0\varphi\,\rangle.\end{aligned}$$

The second part of the assertion follows from the next lemma. ∎

LEMMA 5.1 (Little Regularity Lemma). *Let v be a distribution with $\partial_j v \in L^1_{loc}$ for all $j \in \{1, \ldots, n\}$. Then $v \in L^1_{loc}$.*

Remark 5.1 If $V \in L^p_{loc}$ with $p > 1$ we can use [7, Theorem 4.5.8], and get $v \in L^q_{loc}$, where $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$ if $p < n$, $q \in [\,p, \infty[$ if $p = n$, and $q = \infty$ if $p > n$.

Proof. Let V_j be the locally integrable function associated with the (regular) distribution $\partial_j v$; i.e., $\langle \partial_j v\,, \varphi\,\rangle = \int V_j \varphi$. Define $V_j^{(\ell)} := \eta_\ell * V_j$; hence,

$$\operatorname{curl} V^{(\ell)} = 0 \quad \text{and} \quad V_j^{(\ell)} \xrightarrow{L^1_{loc}} V_j \quad (\ell \to \infty).$$

According to the distributional Poincaré lemma, the distribution $\langle \tilde{v}^{(\ell)}\,, \varphi\,\rangle = \sum_{k=1}^n \langle V_k^{(\ell)}\,, \mu_k \varphi\,\rangle$ is a solution of the equation $\nabla v = V^{(\ell)}$; obviously the C^∞-function $(\eta_\ell * v)$ is another one. Since any two solutions differ by a constant, $\tilde{v}^{(\ell)}$ is in fact a C^∞-function. Now,

$$\int |(\tilde{v}^{(\ell_1)} - \tilde{v}^{(\ell_2)})\varphi| \le \sum_{k=1}^n \int |(V_k^{(\ell_1)} - V_k^{(\ell_2)})\mu_k \varphi| \longrightarrow 0 \quad (\ell_1,\, \ell_2 \to \infty).$$

Hence, $(\tilde{v}^{(\ell)})_{\ell \in \mathbb{N}}$ is a Cauchy sequence in $L^1_{loc}(\mathbb{R}^n)$; denote its limit by $\tilde{v} \in L^1_{loc}(\mathbb{R}^n)$. Again, v and $\tilde{v}$ are distributional solutions of the equation $\nabla v = V$; thus, they differ only by a constant, which implies that v coincides with a locally integrable function. ∎

6. SQUARE INTEGRABLE SOLUTIONS

For reasons explained in the Introduction, we now consider Eq. $(*_n)$ under the additional conditions $u \in L^2(\mathbb{R}^n)$ and $V \in L^p(\mathbb{R}^n)$ for some $p \in [\,1, \infty\,]$. To avoid special cases we assume $n \ge 3$. The spaces $L^{p,\infty} = L^{p,\infty}(\mathbb{R}^n)$ are the well-known weak L^p-spaces.

LEMMA 6.1. *The class* $\{u \in L^{p,\infty} \mid \exists_{v \in L^{q,\infty}} \quad u = e^v\}$ *is empty for every* $(p, q) \in [\,1, \infty\,[\ \times [\,1, \infty\,]$ $(L^{\infty,\infty} = L^\infty)$.

Proof. Assume that there are $u \in L^{p,\infty}$ and $v \in L^{q,\infty}$ such that $u = e^v =: e^{r+is}$. Then[6]

$$\forall_{\tau > 0} \quad C\tau^{-p} \ge m_u(\tau) := \mu_n(\{|u| > \tau\}) \quad \Rightarrow \quad \mu_n(\{|u| \le \tau\}) = \infty \tag{6}$$

and

$$\forall_{\tau > 0} \quad r|_{\{|u| \le \tau\}} = \ln(|u|)|_{\{|u| \le \tau\}} \le \ln(\tau). \tag{7}$$

If $q < \infty$, $\tau = \frac{1}{2}$ and $0 < t < -\ln(\frac{1}{2})$, this implies $m_v(t) \ge \mu_n(\{|r(x)| > t\}) \ge \mu_n(\{|u| \le \frac{1}{2}\}) = \infty$, which is absurd, as $v \in L^{q,\infty}$.

If $q = \infty$ there is a nullset $N \subset \mathbb{R}^n$ and a constant $K > 0$ such that $|r(x)| \le K$ for all $x \in \mathbb{R}^n \backslash N$. Choose $\tau > 0$ such that $\ln(\tau) \le -K$. Then according to (7), $|r(x)| \ge -\ln(\tau) \ge K$ for all $x \in \{|u| \le \tau\}$, which conflicts with our choice of K, since $\{|u| \le \tau\}$ is obviously not a nullset (see (6)). ∎

[6] We write $\{|u| > \tau\}$ for $\{x \in \mathbb{R}^n \mid |u(x)| > \tau\}$; likewise $\{|u| \le \tau\} = \ldots$.

PROPOSITION 6.1. *Let* $p \in \,]1, n[$ *and* $V \in L^p$. *Then Eq.* $(*_n)$ *has no non-trivial solution* $u \in L^2$.

Proof. Let $u \in L^2$ be a solution of $(*_n)$. If u is non-trivial, there exists a AC-lift $v \in AC$ of u which satisfies $\nabla v = V \in L^p$. This implies, according [7, Theorem 4.5.9], the existence of a constant c, s.t. $v - c \in L^q$, where $1/q = 1/p - 1/n$. Hence, $e^{-c}u \in \{\tilde{u} \in L^2 \mid \exists_{\tilde{v} \in L^q} \quad \tilde{u} = e^{\tilde{v}}\}$, which is impossible in view of the previous lemma. ∎

LEMMA 6.2. *There is a constant* $\sigma_n > 0$ *such that Eq.* $(*_n)$ *has no non-trivial solution* $u \in L^2$ *if* $V \in L^{n,\infty}$ *and* $\|V\|_{L^{n,\infty}} < \sigma_n$.

Proof. Let $u \in L^2$ and $V \in L^{n,\infty}$ functions that satisfy $(*_n)$, R_j $(j = 1, \dots, n)$ the Riesz transformations, and I_1 the Riesz potential. From a generalized Hölder inequality (cf. [15]) we get $\nabla u = uV \in L^{q,2}$, where $1/q = 1/2 + 1/n$. It is well known that $f = I_1 \sum_{j=1}^n R_j(\partial_j f)$ (cf. [12, S. 125 (17)]). Hence,

$$u = I_1 \sum_{j=1}^n R_j(\partial_j u) \overset{(*_n)}{=} I_1 \sum_{j=1}^n R_j(uV_j).$$

Since the Riesz transformations and the Riesz potentials are bounded maps $L^{q,2} \to L^{q,2}$ and $L^{q,2} \to L^2$, respectively, we get

$$\|u\|_{L^2} \le C\|uV\|_{L^{q,2}} \le C\|u\|_{L^2}\|V\|_{L^{n,\infty}}.$$

Thus $\|V\|_{L^{n,\infty}} < 1/C =: \sigma_n$ implies $u = 0$. ∎

PROPOSITION 6.2. *Equation* $(*_n)$ *has no non-trivial solution* $u \in L^2$ *if* V *belongs to* L^n.

Proof. Let $u \in L^2$ be a non-trivial solution and $v \in AC$ an AC-lift of u; hence, $\nabla v = V \in L^n \hookrightarrow L^{n,\infty}$. From (4) we get that the function $|u| \in L^2$ is a weak solution of the equation $\nabla|u| - |u|\,R = 0$, where $R := Re(V) \in L^n$. Now, for every $\varepsilon > 0$, we consider the function u_ε, defined by

$$u_\varepsilon(x) := \begin{cases} |u(x)|, & |u(x)| \le \varepsilon \\ \varepsilon, & |u(x)| > \varepsilon. \end{cases}$$

Thus $u_\varepsilon = (|u| - \varepsilon)_- + \varepsilon$ and therefore (cf. [6, Lemma 7.6.])

$$(\nabla u_\varepsilon)(x) = \begin{cases} (\nabla|u|)(x), & |u(x)| \le \varepsilon \\ 0, & |u(x)| > \varepsilon \end{cases}.$$

Set $R_\varepsilon := \chi_{\{|u| \le \varepsilon\}} R \in L^n$, where $\{|u| \le \varepsilon\} := \{x \in \mathbb{R}^n \mid |u(x)| \le \varepsilon\}$. Then $\nabla u_\varepsilon - u_\varepsilon R_\varepsilon = 0$. We shall show below that

$$R_\varepsilon \xrightarrow{L^n} 0 \qquad (\varepsilon \to 0). \tag{8}$$

Therefore there exists an $\varepsilon > 0$ such that $\|R_\varepsilon\|_{L^n} < \sigma_n$, where σ_n denotes the constant appearing in the previous lemma. From this lemma and $\|R_\varepsilon\|_{L^{n,\infty}} \leq \|R_\varepsilon\|_{L^n} < \sigma_n$ we get $u_\varepsilon = 0$, which is obviously absurd.
To show (8) set $U_\varepsilon := \{|u| \leq \varepsilon\}$. Evidently, $U_{\varepsilon_1} \subset U_{\varepsilon_2}$ for all $0 < \varepsilon_1 < \varepsilon_2$. Since u has no zeros outside a certain nullset $N \subset \mathbb{R}^n$, the set $\cap_{\varepsilon>0} U_\varepsilon \subset N$ is a nullset. Therefore, for a.e. $x_0 \in \mathbb{R}^n$ there exists a $\varepsilon_0 > 0$ such that $x_0 \notin U_{\varepsilon_0}$. Hence,

$$\forall_{\varepsilon \leq \varepsilon_0} \qquad R_\varepsilon(x_0) = \chi_{\{|u| \leq \varepsilon\}}(x_0) R(x_0) = 0.$$

This shows that the sequence $(R_\varepsilon)_{\varepsilon>0}$ converges a.e. to zero, as $\varepsilon \to 0$. Now (8) follows from the trivial estimate $|R_\varepsilon| \leq |R| \in L^n$ by Lebesgue's theorem. ∎

Remark 6.1. The set $\{u \in L^2 \mid \exists_{V \in L^{p,\infty}} \nabla u - uV = 0\}$ can be shown to be a dense subset of L^2 for every $p \in [n, \infty]$; cf. [13].

ACKNOWLEDGMENTS

This research was supported in part by the Natural Science and Engineering Research Council of Canada under Grant A 7847. The author acknowledges support by the Pacific Institute of the Mathematical Sciences and thanks the Professors R. Illner and H. Lange for valuable discussions.

REFERENCES

1. G. de Rham, "Differentiable Manifolds," Springer-Verlag, Berlin/Heidelberg/New York, 1984.
2. H.-D. Doebner and G. A. Goldin, On a general nonlinear Schrödinger equation admitting diffusion currents, *Phys. Lett. A* **162** (1992), 397–401.
3. H.-D. Doebner and G. A. Goldin, Properties of nonlinear Schrödinger equations associated with diffeomorphism group representations, *J. Phys. A* **27** (1994), 1771–1780.
4. H.-D. Doebner and G. A. Goldin, Introducing nonlinear gauge transformations in a family of nonlinear Schrödinger equations, *Phys. Rev. A* **54** (1996), 3764–3771.
5. F. G. Friedlander, "Introduction to the Theory of Distributions," Cambridge Univ. Press, Cambridge/New York, 1998.
6. D. Gilbarg and N. S. Trudinger, "Elliptic Partial Differential Equations of Second Order," Springer-Verlag, Berlin/Heidelberg/New York, 1977.
7. L. Hörmander, "The Analysis of Linear Partial Differential Operators, I," Springer-Verlag, Berlin /Heidelberg/New York/Tokyo, 1983.
8. A. Kufner, O. John, and S. Fučík, "Function Spaces," Noorhoff, Leyden, 1977.
9. H. Lange, M. Poppenberg, and H. Teismann, Nonlinear singular Schrödinger type equations, *in* "Nonlinear Theory of Generalized Functions, Proceedings, Workshop, Vienna, Oct.–Dec. 1997" (M. Grosser *et al.,* Eds.), pp. 113–128, Chapman & Hall/CRC, Boca Raton, 1999.

10. M. Oberguggenberger, "Multiplication of Distributions and Applications to Partial Differential Equations," Longman, Harlow, 1992.
11. H. Schubert, "Topology," Allyn & Bacon, Boston, 1968.
12. E. U. Stein, "Singular Integrals and Differentiability Properties of Functions," Princeton Univ. Press, Princeton, NJ, 1970.
13. H. Teismann, "Das Cauchy–Problem für die Doebner–Goldin–Gleichung," Ph.D. thesis, Universität zu Köln, 1997.
14. H. Teismann, The Cauchy problem for the Doebner-Goldin equation, *in* "GROUP 21—Physical Applications and Mathematical Aspects of Geometry, Groups and Algebras" (H.-D. Doebner *et al.,* Eds.), Vol. 1, pp. 433–438. World Scientific, Singapore, 1997.
15. H. Teismann. The Schrödinger equation with singular time-dependent potentials, *Discrete Cont. Dynam. Systems* **6** (2000), 705–722.
16. W. P. Ziemer, "Weakly Differentiable Functions: Sobolev Spaces and Functions of Bounded Variation," Springer-Verlag, New York, 1989.